

Further extension of Nadler's fixed point theorem

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Abstract. In this paper, we prove a generalization of Geraghty's fixed point theorem for multi-valued mappings.

1. INTRODUCTION

Many fixed point theorems have been proved by various authors as generalizations to Banach's contraction mapping principle. One such generalization is due to Geraghty [3] as follows.

Theorem 1.1. *Let (X, d) be a complete metric space, let $f : X \rightarrow X$ be a mapping such that for each $x, y \in X$,*

$$d(f(x), f(y)) \leq \alpha(d(x, y)) d(x, y)$$

where α is a function from $[0, \infty)$ into $[0, 1)$ which satisfy the simple condition $\alpha(t_n) \rightarrow 1 \implies t_n \rightarrow 0$. Then f has a fixed point $z \in X$, and $\{f^n(x)\}$ converges to z , for each $x \in X$.

Let (X, d) be a metric space. Let $CB(X)$ denotes the collection of all nonempty closed bounded subsets of X . For $A, B \in CB(X)$ and $x \in X$, define $D(x, A) := \inf\{d(x, a); a \in A\}$ and

$$H_d(A, B) := \max\{\sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A)\}.$$

It is easy to see that H_d is a metric on $CB(X)$. H_d is called the Hausdorff metric induced by d . A point $p \in X$ is said to be a fixed point of multi-valued mapping $T : X \rightarrow CB(X)$ if $p \in T(p)$.

The fixed point theory of multi-valued contractions was initiated by Nadler [5] in the following way.

Theorem 1.2. (Nadler [5].) *Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$ such that for all $x, y \in X$,*

$$H_d(Tx, Ty) \leq r d(x, y) \tag{1}$$

where, $0 \leq r < 1$. Then T has a fixed point.

This theory was developed in different directions by many authors, in particular, by Mizoguchi and Takahashi [4].

Theorem 1.3. (Mizoguchi and Takahashi [4].) *Let (X, d) be a complete metric space and let T be a mapping from X to $CB(X)$. Assume*

$$H_d(Tx, Ty) \leq \alpha(d(x, y)) d(x, y) \tag{2}$$

for all $x, y \in X$, where α is a function from $[0, \infty)$ into $[0, 1)$ satisfying $\limsup_{s \rightarrow t+} \alpha(s) < 1$ for all $t \in [0, \infty)$. Then T has a fixed point.

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Recently, Eldred et al. [2] claimed that Nadler's fixed point theorem is equivalent to Mizoguchi and Takahashi's fixed point theorem. Very recently, Suzuki [7] produced an example to disprove their claim and showed that Mizoguchi and Takahashi's fixed point theorem is a real generalization of Nadler's theorem.

In this paper, we extended the Geraghty's fixed point theorem to multi-valued mappings. Also we give an example to show that our theorem is a real generalization of Nadler's.

2. MAIN RESULT

Let S denotes the class of those functions $\alpha : [0, \infty) \rightarrow [0, 1)$ which satisfy the simple condition $\alpha(t_n) \rightarrow 1 \implies t_n \rightarrow 0$.

Theorem 2.1. *Let (X, d) be a complete metric space, let $T : X \rightarrow CB(X)$, and suppose there exists $\alpha \in S$ such that for each $x, y \in X$*

$$H_d(Tx, Ty) \leq \alpha(d(x, y)) d(x, y). \quad (3)$$

Then T has a fixed point.

Proof: Define a function β from $[0, \infty)$ into $[0, 1)$ by

$$\beta(t) := \frac{1 + \alpha(t)}{2}.$$

Then the following hold:

- 1) $\alpha(t) < \beta(t)$ for all t ,
- 2) $\beta \in S$.

Let $x_0 \in X$ be arbitrary and fixed and let $x_1 \in Tx_0$. If $x_1 = x_0$, then x_0 is a fixed point of T , and the proof is complete. Now, let $x_1 \neq x_0$. Then we have

$$D(x_1, Tx_1) \leq H_d(Tx_0, Tx_1) \leq \alpha(d(x_0, x_1)) d(x_0, x_1) < \beta(d(x_0, x_1)) d(x_0, x_1).$$

Thus there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq \beta(d(x_0, x_1)) d(x_0, x_1).$$

Now, if $x_1 = x_2$, then x_1 is a fixed point of T , and the proof is complete. We suppose that $x_1 \neq x_2$. Then

$$D(x_2, Tx_2) \leq H_d(Tx_1, Tx_2) \leq \alpha(d(x_1, x_2)) d(x_1, x_2) < \beta(d(x_1, x_2)) d(x_1, x_2).$$

Hence, there exists $x_3 \in Tx_2$ satisfying

$$d(x_2, x_3) \leq \beta(d(x_1, x_2)) d(x_1, x_2).$$

Inductively, for each positive integer number n , there exists $x_{n+1} \in Tx_n$, $x_{n+1} \neq x_n$, satisfying

$$d(x_n, x_{n+1}) \leq \beta(d(x_{n-1}, x_n)) d(x_{n-1}, x_n).$$

To show that $\{x_n\}$ is a Cauchy sequence, we break the argument into two steps.

Step1. $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$.

Proof. Since $\beta(t) < 1$ for all t , $\{d(x_n, x_{n+1})\}$ is decreasing and bounded below, so

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = r \geq 0.$$

Assume $r > 0$. Then we have

$$\frac{d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1})} \leq \beta(d(x_n, x_{n+1})) , \quad n = 1, 2, \dots .$$

Letting $n \rightarrow \infty$ we see that $1 \leq \lim_{n \rightarrow \infty} \beta(d(x_n, x_{n+1}))$, and since $\beta \in S$ this in turn implies $r = 0$. This contradiction established Step 1.

Step 2. $\{x_n\}$ is a Cauchy sequence.

Proof. Assume $\limsup_{n, m \rightarrow \infty} d(x_n, x_m) > 0$. By triangle inequality for positive real numbers n, m and for $y \in Tx_n$, we obtain

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, y) + d(y, x_{m+1}) + d(x_{m+1}, x_m).$$

This means that for every positive real numbers m, n ,

$$\begin{aligned} d(x_n, x_m) &\leq D(x_{n+1}, Tx_n) + D(x_{m+1}, Tx_n) + d(x_n, x_{n+1}) + d(x_m, x_{m+1}) \\ &\leq H_d(Tx_m, Tx_n) + d(x_n, x_{n+1}) + d(x_m, x_{m+1}) \\ &\leq \beta(d(x_m, x_n)) d(x_n, x_m) + d(x_n, x_{n+1}) + d(x_m, x_{m+1}). \end{aligned}$$

Hence,

$$d(x_n, x_m) \leq (1 - \beta(d(x_n, x_m)))^{-1} (d(x_n, x_{n+1}) + d(x_m, x_{m+1})).$$

Under the assumption $\limsup_{n, m \rightarrow \infty} d(x_n, x_m) > 0$, Step 1 now implies that

$$\limsup_{n, m \rightarrow \infty} \frac{1}{1 - \beta(d(x_n, x_m))} = +\infty$$

for which

$$\limsup_{n, m \rightarrow \infty} \beta(d(x_n, x_m)) = 1.$$

On the other hand we have $\beta \in S$. It follows that $\limsup_{n, m \rightarrow \infty} d(x_n, x_m) = 0$ which is a contradiction.

Now, we will complete the proof by observing that $\{x_n\}$ is Cauchy sequence. By completeness of X , there exists $z \in X$ such that $\lim_{n \rightarrow \infty} x_n = z$. Since T is continuous, then $\lim_{n \rightarrow \infty} Tx_n = Tz$. Hence,

$$\begin{aligned} D(z, Tz) &= D(\lim_{n \rightarrow \infty} x_{n+1}, Tz) = \lim_{n \rightarrow \infty} D(x_{n+1}, Tz) \\ &\leq \lim_{n \rightarrow \infty} H_d(Tx_n, Tz) \\ &\leq \lim_{n \rightarrow \infty} \beta(d(x_n, z)) d(x_n, z) \\ &\leq \lim_{n \rightarrow \infty} d(x_n, z) = 0. \end{aligned}$$

On the other hand Tz is closed. Then $z \in Tz$. □

The following example shows that Theorem 2.1 is a real generalization of Nadler's.

Example 2.2. Let l^∞ be the Banach space consisting of all bounded real sequences with supremum norm and let $\{e_n\}$ be a canonical basis of l^∞ . Let $\{\tau_n\}$ be a bounded, strictly decreasing sequence in $(0, 1)$ such that $\tau_1 = \frac{1}{2}$ and for each positive integer number n , $\tau_{n+1} = (1 - \tau_n)\tau_n$. It is easy to see that $\tau_n \downarrow 0$. Put $x_n = \tau_n e_n$ and $X_n = \{x_n, x_{n+1}, \dots\}$ for all $n \in \mathbb{N}$. Define a bounded, complete subset X of l^∞ by $X = X_1$. Now define a map $T : X \rightarrow CB(X)$ as

$$Tx_n = X_{n+1}, \quad (n \in \mathbb{N})$$

and $\alpha : [0, \infty) \rightarrow [0, 1)$ as

$$\alpha(t) = \begin{cases} 1 - \tau_n & t = \tau_n \ (n \in \mathbb{N}), \\ 0 & \text{otherwise.} \end{cases}$$

Now we can see that:

- (i) T satisfies (3) for all $x, y \in X$.
- (ii) There is no T -invariant subset M such that $M \neq \emptyset$ and (1) holds for all $x, y \in M$.
- (iii) If $\alpha(t_n) \rightarrow 1$, then $t_n \rightarrow 0$.

proof. It is easy to see that the following hold:

- For $m > n$, $H(Tx_m, Tx_n) = \tau_{n+1}$.
- For $m > n$, $d(x_m, x_n) = \tau_n$.

Fix $m, n \in \mathbb{N}$ with $m > n$, we have

$$H(Tx_m, Tx_n) = \tau_{n+1} = (1 - \tau_n) \tau_n = \alpha(d(x_m, x_n)) d(x_m, x_n).$$

It follows (i). We note that X_n 's are only T -invariant subsets of X because fix $n \in \mathbb{N}$, for each $k \geq n$, $Tx_k = X_{k+1} \subseteq X_n$. If for some $k \in \mathbb{N}$, (1) holds for all $x, y \in X_k$, then there exists $r \in (0, 1)$ such that for each $m > n \geq k$

$$H(Tx_m, Tx_n) \leq r d(x_m, x_n).$$

Hence, for all $n \geq k$ we obtain

$$\tau_n (1 - \tau_n) \leq r \tau_n.$$

this shows that for each $n \geq k$, $1 - r \leq \tau_n$. This is contradiction with $\lim_{n \rightarrow \infty} \tau_n = 0$. Thus, we obtain (ii). It is easy to see that (iii) holds. On the other hand we have

$$\limsup_{t \rightarrow 0^+} \alpha(t) = \limsup_{n \rightarrow \infty} (1 - \tau_n) = 1.$$

This shows that α does not satisfy in conditions of Mizoguchi and Takahashi's theorem.

Let (X, d) be a metric space. Let $k \in \mathbb{N}$, and $\mathcal{M}^0 := X, \mathcal{H}^0 := d$, for each $i \in \{1, 2, \dots, k\}$, put $\mathcal{M}^i := CB(\mathcal{M}^{i-1})$ and $\mathcal{H}^i := H_{\mathcal{H}^{i-1}}$. One can show that $(\mathcal{M}^i, \mathcal{H}^i)$ is a complete metric space for all $i \in \{1, 2, \dots, k\}$, whenever (X, d) is a complete metric space (see for example Lemma 8.1.4, of [6]). Every mapping T from X into \mathcal{M}^k is called generalized multi-valued mapping. Recently, M. Eshaghi Gordji et al. [1] proved a generalized multi-valued extension of Nadler's fixed point theorem. The question arises here is whether Theorem 2.1 can be extended to generalized multi-valued mappings or not?

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